Exercise 76

Suppose the curve $y = x^4 + ax^3 + bx^2 + cx + d$ has a tangent line when $x = 0$ with equation $y = 2x + 1$ and a tangent line when $x = 1$ with equation $y = 2 - 3x$. Find the values of a, b, c, and d.

Solution

Take the derivative of the equation for the curve.

$$\begin{aligned} y' &= \frac{d}{dx}(x^4 + ax^3 + bx^2 + cx + d) \\ &= \frac{d}{dx}(x^4) + \frac{d}{dx}(ax^3) + \frac{d}{dx}(bx^2) + \frac{d}{dx}(cx) + \frac{d}{dx}(d) \\ &= \frac{d}{dx}(x^4) + a\frac{d}{dx}(x^3) + b\frac{d}{dx}(x^2) + c\frac{d}{dx}(x) + \frac{d}{dx}(d) \\ &= (4x^3) + a(3x^2) + b(2x) + c(1) + (0) \\ &= 4x^3 + 3ax^2 + 2bx + c \end{aligned}$$

Use the fact that at $x = 0$, the slope of the curve is 2.

$$y'(0) = c = 2 \tag{1}$$

Use the fact that at $x = 1$, the slope of the curve is -3.

$$y'(1) = 4 + 3a + 2b + c = -3 \tag{2}$$

Also, use the fact that at $x = 0$, $y = 1$.

$$y(0) = (0)^4 + a(0)^3 + b(0)^2 + c(0) + d = d = 1 \tag{3}$$

Also, use the fact that at $x = 1$, $y = -1$.

$$y(1) = (1)^4 + a(1)^3 + b(1)^2 + c(1) + d = 1 + a + b + c + d = -1 \tag{4}$$

Solve equations (1), (2), (3), and (4) for a, b, c, and d.

$$a = 1 \qquad b = -6 \qquad c = 2 \qquad d = 1$$

Therefore, the curve is

$$y = x^4 + x^3 - 6x^2 + 2x + 1.$$

Below is a graph of the curve with the two tangent lines at $x = 0$ and $x = 1$.